

INVERTIBLE CARNOT GROUPS

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ABSTRACT. We characterize Carnot groups admitting a 1-quasiconformal metric inversion as the Lie groups of Heisenberg type whose Lie algebras satisfy the J^2 -condition, thus characterizing a special case of inversion invariant bi-Lipschitz homogeneity. A more general characterization of inversion invariant bi-Lipschitz homogeneity for certain non-fractal metric spaces is also provided.

1. INTRODUCTION

In [CDKR91, Theorem 5.1], the authors characterize the nilpotent components of Iwasawa decompositions of real rank one simple Lie groups by the fact that they are of Heisenberg type and admit certain conformal inversions. The conformal inversions of [CDKR91, Theorem 5.1] generalize the usual Möbius inversions of Euclidean space. Both of these notions of inversion are generalized in turn by the concept of *metric inversion* as studied in [BHX08]. In the present paper, we study Carnot groups that admit such metric inversions.

The Carnot groups of [CDKR91] are equipped with a left-invariant gauge distance that is bi-Lipschitz equivalent to a left-invariant sub-Riemannian (or more generally, sub-Finsler) distance. The study of such left-invariant distances can be generalized by the study of metric spaces that admit a transitive family of uniformly bi-Lipschitz self-homeomorphisms. These spaces are said to be *uniformly bi-Lipschitz homogeneous*.

The concepts of metric inversion and bi-Lipschitz homogeneity are combined in the notion of *inversion invariant bi-Lipschitz homogeneity* as studied in [Fre12] and [Fre13]. In particular, [Fre13, Theorem 2.4] demonstrates that certain inversion invariant bi-Lipschitz homogeneous geodesic spaces are bi-Lipschitz equivalent to Carnot groups (cf. [LD11a], [LD11b]). However, [Fre13, Theorem 2.4] does not provide a characterization of Carnot groups via inversion invariant bi-Lipschitz homogeneity. Indeed, in light of [CDKR91, Theorem 5.1], the invertibility of a Carnot group appears to be a very restrictive condition. The following theorem identifies the Carnot groups admitting a metric inversion under the additional assumption that the inversion is 1-quasiconformal (see Section 2 and Section 3 for relevant definitions).

Theorem 1.1. *Suppose G is a sub-Riemannian Carnot group. The group G admits a 1-quasiconformal metric inversion if and only if it is isomorphic to a generalized Heisenberg group.*

While the assumption of 1-quasiconformality is strong it is not altogether unnatural. Indeed, the inversions of [CDKR91, Theorem 5.1] are 1-quasiconformal. Furthermore, Theorem 1.1 illustrates a special case of the characterization of conformal maps on Carnot groups recently provided by [CO13, Theorem 4.1]. One should note, however, that the methods used to prove Theorem 1.1 are somewhat different from those implemented in [CO13].

In general, metric inversions are K -quasiconformal, with $1 \leq K < +\infty$ (see Section 2). One can obtain an analogue of Theorem 1.1 for general metric inversions if more information is assumed about the Carnot groups in question. In particular, the following fact is contained in recent work of Xiangdong Xie (see [Xie13b], [Xie13c], and [Xie13a] for relevant definitions and results).

Date: August 21, 2014.

2010 Mathematics Subject Classification. 53C17, 30L10, 30L05.

Key words and phrases. metric inversion, bi-Lipschitz homogeneity, Carnot groups.

Fact 1.2. *Suppose \mathbb{G} is a non-rigid sub-Riemannian Carnot group. If \mathbb{G} is not contained in one of the following three classes of groups, then it does not admit a metric inversion:*

- (1) *Euclidean groups*
- (2) *Heisenberg product groups*
- (3) *Complex Heisenberg product groups*

Let \mathbb{G} be a non-rigid Carnot group that is not contained in one of the three classes listed above. Let $\hat{\mathbb{G}}$ denote a metric sphericalization of \mathbb{G} (see Section 2). It follows from the aforementioned work of Xie that any quasiconformal self-homeomorphism $f : \hat{\mathbb{G}} \rightarrow \hat{\mathbb{G}}$ must permute left cosets of certain proper subgroups of \mathbb{G} , and therefore must fix the point at infinity. Fact 1.2 then follows from the observation that metric inversions do not fix the point at infinity.

While Theorem 1.1 falls within the general study of inversion invariant bi-Lipschitz homogeneity, it exhibits a special case of this property in the particular setting of Carnot groups. Note that (non-abelian) Carnot groups are *fractal* in the sense that their topological and Hausdorff dimensions do not agree. When we restrict ourselves to non-fractal metric spaces, we can use recent work of Kinneburg (see [Kin13]) to obtain the following result.

Theorem 1.3. *Suppose X is a proper and connected metric space whose Hausdorff and topological dimensions are both equal to $n \in \mathbb{N}$. The space X is inversion invariant bi-Lipschitz homogeneous if and only if it is bi-Lipschitz equivalent to \mathbb{R}^n (when X is unbounded) or \mathbb{S}^n (when X is bounded).*

In fact, an analogue of Theorem 1.3 holds when the Hausdorff dimension of X is strictly larger than its topological dimension, but this requires additional assumptions about the one-dimensional metric structure of X (cf. [Kin13, Theorem 1.5]).

Acknowledgments. The proof of Theorem 1.1 was inspired by observations made in [BS13, Section 6.3] about the Tits classification of 2-transitive group actions. The author is grateful to Xiangdong Xie for many helpful discussions during the preparation of this paper, and for the helpful suggestions of the anonymous referees.

2. NOTATION AND DEFINITIONS

Given two numbers A and B , we write $A \simeq_C B$ to indicate that $C^{-1}A \leq B \leq CA$, where C is independent of A and B . When the quantity C is understood or irrelevant, we simply write $A \simeq B$.

Given a metric space (X, d) , $r > 0$, and $x \in X$ we write $B(x; r) := \{y \in X : d(x, y) < r\}$ to denote an open ball in X centered at x of radius r . A metric space X is said to be *proper* if the closures of open balls in X are compact.

Given $1 \leq L < +\infty$, an embedding $f : X \rightarrow Y$ is *L-bi-Lipschitz* provided that for all points $x, y \in X$ we have $d_Y(f(x), f(y)) \simeq_L d_X(x, y)$. Two spaces X and Y are *L-bi-Lipschitz equivalent* if there exists an *L-bi-Lipschitz* homeomorphism between the two spaces. A space X is *bi-Lipschitz homogeneous* if there exists a collection \mathcal{F} of bi-Lipschitz self-homeomorphisms of X such that, for every pair $x, y \in X$, there exists $f \in \mathcal{F}$ with $f(x) = y$. When every map in \mathcal{F} is *L-bi-Lipschitz* we say that X is *L-bi-Lipschitz homogeneous*, or *uniformly bi-Lipschitz homogeneous* when the particular distortion bound is not important.

An embedding $f : X \rightarrow Y$ is *θ -quasimöbius* if $\theta : [0, +\infty) \rightarrow [0, +\infty)$ is a homeomorphism and, for all quadruples x, y, z, w of distinct points in X , we have

$$\frac{d(f(x), f(y))d(f(z), f(w))}{d(f(x), f(z))d(f(y), f(w))} \leq \theta \left(\frac{d(x, y)d(z, w)}{d(x, z)d(y, w)} \right).$$

When there exists a constant $1 \leq C < +\infty$ such that $\theta(t) = Ct$, we say that f is *strongly quasimöbius*. See [Kin13] for a detailed study of such mappings.

An embedding $f : X \rightarrow Y$ is *metrically K-quasiconformal* if $1 \leq K < +\infty$ and, for all $x \in X$,

$$\limsup_{r \rightarrow 0} \frac{\sup\{d(f(x), f(y)) : d(x, y) \leq r\}}{\inf\{d(f(x), f(z)) : d(x, z) \geq r\}} \leq K.$$

In the present paper, since no other definition of quasiconformality is used, we drop the qualifier ‘metrically,’ and simply refer to K -quasiconformal mappings.

Given an unbounded metric space (X, d) and a point $p \in X$, we define $\hat{X} := X \cup \{\infty\}$ and $X_p := X \setminus \{p\}$. We say that a homeomorphism $\varphi : X_p \rightarrow X_p$ is an L -metric inversion provided that there exists a constant $1 \leq L < +\infty$ such that for any $x, y \in X_p$,

$$d(\varphi(x), \varphi(y)) \simeq_L \frac{d(x, y)}{d(x, p)d(y, p)}.$$

We extend φ to \hat{X} by the definitions $\varphi(\infty) := p$ and $\varphi(p) := \infty$. We say that (X, d) admits a metric inversion provided that there exists a point $p \in X$ such that an L -metric inversion exists on X_p . We note that the inversions of [CDKR91, Theorem 4.2] are 1-metric inversions. In general, while L -metric inversions are L^2 -quasiconformal, they need not be 1-quasiconformal.

The definition of metric inversion given above is closely related to the metric inversion described in [BHX08]. In [BHX08], the authors construct a distance d_p on \hat{X}_p such that, for any $x, y \in X_p$,

$$\frac{1}{4} \cdot \frac{d(x, y)}{d(x, p)d(y, p)} \leq d_p(x, y) \leq \frac{d(x, y)}{d(x, p)d(y, p)},$$

with obvious analogues when x or y is equal to ∞ . They also prove that the identity map from (X_p, d) to (X_p, d_p) is $16t$ -quasimöbius ([BHX08, Lemma 3.1]) and 1-quasiconformal at non-isolated points ([BHX08, Proposition 4.1]). While the [BHX08] definition of metric inversion is valid for both bounded and unbounded spaces, if X is an unbounded space admitting an L -metric inversion φ at p , then $\varphi : (X_p, d_p) \rightarrow (X_p, d)$ is a $4L$ -bi-Lipschitz homeomorphism.

A related concept is that of *metric sphericalization*. Again following [BHX08], the metric sphericalization of an unbounded metric space (X, d) at some basepoint $p \in X$ is denoted by (\hat{X}, \hat{d}_p) . Here \hat{d}_p is a distance such that, for any $x, y \in \hat{X}$,

$$(2.1) \quad \frac{1}{4} \cdot \frac{d(x, y)}{(1 + d(x, p))(1 + d(y, p))} \leq \hat{d}_p(x, y) \leq \frac{d(x, y)}{(1 + d(x, p))(1 + d(y, p))}.$$

As with metric inversion (see [BHX08, Section 3.B]), the identity map from (X, d) to (\hat{X}, \hat{d}_p) is $16t$ -quasimöbius and 1-quasiconformal at non-isolated points.

Following [Fre12] and [Fre13], a metric space (X, d) is *inversion invariant bi-Lipschitz homogeneous* provided that both (X, d) and (\hat{X}_p, d_p) are uniformly bi-Lipschitz homogeneous. This definition is independent of $p \in X$, up to a quantitative change in parameters. One can verify that, when (X, d) is unbounded, this definition is equivalent to the statement that (X, d) is bi-Lipschitz homogeneous and (X, d) admits a metric inversion.

Given a rectifiably connected metric measure space and $1 \leq Q < +\infty$, we say that X is a *Loewner space* if there exists a non-increasing function $\eta : (0, +\infty) \rightarrow (0, +\infty)$ such that $\text{Mod}_Q(E, F) \geq \eta(t)$ for any disjoint nondegenerate continua $E, F \subset X$. Here $t \geq \text{dist}(E, F) / \min\{\text{diam}(E), \text{diam}(F)\}$ and $\text{Mod}_Q(E, F)$ denotes the *conformal modulus* of the family of all curves joining E to F in X (see, for example, [HK98, Section 2]).

The Loewner condition is often examined in the context of *Ahlfors regular* metric measure spaces. Given $0 < Q < +\infty$, a metric space (X, d) with Borel measure μ is Ahlfors Q -regular provided that there exists a constant $1 \leq C < +\infty$ such that $\mu(B(x; r)) \simeq_C r^Q$ for every $x \in X$ and $0 < r \leq \text{diam}(X)$.

Carnot groups are examples of Ahlfors Q -regular Loewner spaces. A Carnot group \mathbb{G} of step $n \in \mathbb{N}$ is a connected, simply connected, nilpotent Lie group with stratified Lie algebra $\text{Lie}(\mathbb{G}) = V_1 \oplus V_2 \oplus \cdots \oplus V_n$. The *layers* V_i are such that, for $1 \leq j \leq n-1$, we have $[V_j, V_1] = V_{j+1}$. Here $[X, Y] = XY - YX$ denotes the Lie bracket. We require $V_n \neq \{0\}$, and that for each $1 \leq j \leq n$ we have $[V_j, V_n] = \{0\}$. We refer to V_1 as the *horizontal layer* of $\text{Lie}(\mathbb{G})$. By left-translation V_1 is extended to a (left-invariant) distribution Δ on \mathbb{G} , referred to as the *horizontal distribution*.

When Δ is equipped with a left-invariant norm $|\cdot|$, we define the associated *sub-Finsler distance* d_{SF} on \mathbb{G} as follows. Let $\gamma : [0, 1] \rightarrow \mathbb{G}$ be an absolutely continuous path. The path γ is *horizontal* provided that for almost every $t \in [0, 1]$ we have $\dot{\gamma}(t) \in \Delta$. The d_{SF} length of a horizontal path γ is $\ell_{SF}(\gamma) := \int_0^1 |\dot{\gamma}(t)| dt$. We then define

$$d_{SF}(x, y) := \inf\{\ell_{SF}(\gamma) : \gamma \text{ a horizontal path such that } \gamma(0) = x, \gamma(1) = y\}.$$

By well known results of Chow and Rashevskii, d_{SF} defines a geodesic distance on \mathbb{G} . Thus a *sub-Finsler Carnot group* is a Carnot group equipped with a distance d_{SF} . When the norm on Δ is derived from an inner product, we obtain a *sub-Riemannian distance* on \mathbb{G} , denoted by d_{SR} .

Since the norm on Δ is left-invariant, for any element $g \in \mathbb{G}$, the left translation $\ell_g(x) := gx$ is an isometry of (\mathbb{G}, d_{SF}) . Sub-Finsler distance is also homogeneous with respect to *canonical dilations*. For an element $x \in \mathbb{G}$, let $X := \log(x) = \sum_{i=1}^n X_i$, where $X_i \in V_i$. For any $t > 0$, define the canonical dilation $\delta_t : (\mathbb{G}, d_{SF}) \rightarrow (\mathbb{G}, d_{SF})$ as $\delta_t(x) = \exp(\sum_{i=1}^n t^i X_i)$. For points $x, y \in \mathbb{G}$, we have $d_{SF}(\delta_t(x), \delta_t(y)) = t d_{SF}(x, y)$.

3. GENERALIZED HEISENBERG GROUPS

Let \mathfrak{n} denote a Lie algebra endowed with an inner product $\langle \cdot, \cdot \rangle$ and accompanying norm $|\cdot|$. Suppose that \mathfrak{n} is either abelian or stratified of step two. In the step two case, this means that there exist non-trivial complementary orthogonal subspaces \mathfrak{v} and \mathfrak{z} such that $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$ and $[\mathfrak{v}, \mathfrak{z}] = \{0\} = [\mathfrak{z}, \mathfrak{z}]$. For $X, Y \in \mathfrak{v}$ and $Z \in \mathfrak{z}$, let the map $J : \mathfrak{z} \rightarrow \text{End}(\mathfrak{v})$ be defined via the formula $\langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle$. The algebra \mathfrak{n} is of *Heisenberg type* provided that, for all $X \in \mathfrak{v}$ and $Z \in \mathfrak{z}$, we have $|J_Z X| = |Z||X|$. Equivalently, $J_Z^2 = -|Z|^2 I$, where I denotes the identity map. Various properties of the map J are documented in [CDKR98, Section 2(a)]. A simply connected Lie group is said to be of Heisenberg type if its Lie algebra is of Heisenberg type.

Given a Lie algebra \mathfrak{n} of Heisenberg type, we say that \mathfrak{n} satisfies the J^2 -condition provided that, for any $X \in \mathfrak{v}$ and any two orthogonal elements $Z, Z' \in \mathfrak{z}$, there exists some element $Z'' \in \mathfrak{z}$ such that $J_Z J_{Z'} X = J_{Z''} X$. Note that if $\dim(\mathfrak{z}) \in \{0, 1\}$, this condition is vacuously satisfied.

We say that a Carnot group is a *generalized Heisenberg group* if it is a Heisenberg group over \mathbb{K} , where (here and in the sequel) \mathbb{K} denotes either the real numbers \mathbb{R} , complex numbers \mathbb{C} , quaternions \mathbb{H} , or octonions \mathbb{O} . These groups are defined as follows:

- The Heisenberg group over \mathbb{R} , or the *real Heisenberg group* $H_{\mathbb{R}}$, is \mathbb{R}^n .
- The Heisenberg group over \mathbb{C} , or the *complex Heisenberg group* $H_{\mathbb{C}}$, is the Carnot group with step two real Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{v} := \text{Span}\{X_i, Y_i : 1 \leq i \leq n\}$ and $\mathfrak{z} := \text{Span}\{Z\}$. Equip \mathfrak{n} with an inner product such that $\{X_i, Y_i, Z : 1 \leq i \leq n\}$ is an orthonormal basis. The only non-trivial bracket relations are $[X_i, Y_i] = Z$, for $1 \leq i \leq n$.
- The Heisenberg group over \mathbb{H} , or the *quaternionic Heisenberg group* $H_{\mathbb{H}}$, is the Carnot group with step two real Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{v} = \text{Span}\{X_i, Y_i, V_i, W_i : 1 \leq i \leq n\}$ and $\mathfrak{z} = \text{Span}\{Z_k : 1 \leq k \leq 3\}$. Equip \mathfrak{n} with an inner product such that $\{X_i, Y_i, V_i, W_i, Z_k : 1 \leq i \leq n, 1 \leq k \leq 3\}$ is an orthonormal basis. For $1 \leq i \leq n$, the only nontrivial bracket relations are $[X_i, Y_i] = Z_1 = [V_i, W_i]$, $[X_i, V_i] = Z_2 = [W_i, Y_i]$, and $[X_i, W_i] = Z_3 = [Y_i, V_i]$.
- The Heisenberg group over \mathbb{O} , or the *octonionic Heisenberg group* $H_{\mathbb{O}}$, is the Carnot group with step two real Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{v} = \text{Span}\{X_i : 0 \leq i \leq 7\}$ and $\mathfrak{z} = \text{Span}\{Z_k : 1 \leq k \leq 7\}$. Equip \mathfrak{n} with an inner product such that $\{X_i, Z_k : 0 \leq i \leq 7, 1 \leq k \leq 7\}$ is an orthonormal basis. The only nontrivial bracket relations are $[X_0, X_k] = Z_k$ for $1 \leq k \leq 7$ and $[X_i, X_j] = \varepsilon_{ijk} Z_k$, for $1 \leq i, j, k \leq 7$. Here ε is a completely antisymmetric tensor whose value is $+1$ when $ijk = 124, 137, 156, 235, 267, 346, 457$.

Via exponential coordinates, parameterizations of the groups $H_{\mathbb{K}}$ can be obtained as follows:

- When $\mathbb{K} = \mathbb{R}$, the abelian group $H_{\mathbb{K}}$ is equal to \mathbb{R}^n .

- When $\mathbb{K} = \mathbb{C}$, for each $1 \leq i \leq n$, identify $x_i X_i + y_i Y_i$ with $x_i e_0 + y_i e_1 \in \mathbb{C}$. Here $\{e_0, e_1\}$ is the canonical basis over \mathbb{R} for \mathbb{C} . Thus $\text{Span}\{X_i, Y_i : 1 \leq i \leq n\}$ is identified with \mathbb{C}^n . Identify zZ with $ze_1 \in \mathfrak{S}(\mathbb{C})$.
- When $\mathbb{K} = \mathbb{H}$, for each $1 \leq i \leq n$, identify $x_i X_i + y_i Y_i + v_i V_i + w_i W_i$ with $x_i e_0 + y_i e_1 + v_i e_2 + w_i e_3$. Here $\{e_i\}_{i=0}^3$ is the canonical basis over \mathbb{R} for \mathbb{H} . Thus $\text{Span}\{X_i, Y_i, V_i, W_i : 1 \leq i \leq n\}$ is identified with \mathbb{H}^n . For each $1 \leq k \leq 3$, identify $z_k Z_k$ with $z_k e_k \in \mathfrak{S}(\mathbb{H})$.
- When $\mathbb{K} = \mathbb{O}$, identify $\sum_{i=0}^7 x_i X_i$ with $\sum_{i=0}^7 x_i e_i$ and $\sum_{k=1}^7 z_k Z_k$ with $\sum_{k=1}^7 z_k e_k$. Here $\{e_i\}_{i=0}^7$ is the canonical basis over \mathbb{R} for \mathbb{O} . Thus $\text{Span}\{X_i : 0 \leq i \leq 7\}$ is identified with \mathbb{O} and $\text{Span}\{Z_k : 1 \leq k \leq 7\}$ is identified with $\mathfrak{S}(\mathbb{O})$.

Extending the above identifications by linearity allows us to parameterize $H_{\mathbb{K}}$ by $\mathbb{K}^n \oplus \mathfrak{S}(\mathbb{K})$. When $\mathbb{K} = \mathbb{R}$ we have $\mathfrak{S}(\mathbb{K}) = \{0\}$, and when $\mathbb{K} = \mathbb{O}$ we have $n = 1$. Let $(x, z) = (\sum_{i=1}^n x_i, z)$ denote a point in $H_{\mathbb{K}} = \mathbb{K}^n \oplus \mathfrak{S}(\mathbb{K})$. Via the Baker-Campbell-Hausdorff formula, for points $(x, z), (x', z') \in H_{\mathbb{K}}$, the group law reads as

$$(x, z)(x', z') = \left(x + x', z + z' - \frac{1}{2} \sum_{i=1}^n \mathfrak{S}(x_i \overline{x'_i}) \right).$$

Let $m = \dim(\mathfrak{S}(\mathbb{K}))$, so that $m \in \{0, 1, 3, 7\}$. Given a canonical basis element $e_k \in \mathfrak{S}(\mathbb{K})$, we define a map $L_k \in \text{End}(\mathbb{K}^n)$ such that, for $x \in \mathbb{K}^n$, $L_k(x) = e_k x = \sum_{i=0}^n e_k x_i$. In other words, L_k denotes left multiplication in \mathbb{K}^n by e_k . Passing through the above parameterization of $H_{\mathbb{K}}$ and extending by linearity, this gives rise to a map $J : \mathfrak{z} \rightarrow \text{End}(\mathfrak{v})$ such that, for any $X, Y \in \mathfrak{v}$ and $Z \in \mathfrak{z}$, we have $\langle J_Z X, Y \rangle = \langle [X, Y], Z \rangle$. Furthermore, it is straightforward to verify that, for any $Z \in \mathfrak{z}$, the map $J_Z : \mathfrak{v} \rightarrow \mathfrak{v}$ satisfies $J_Z^2 = -|Z|^2 I$. For two orthogonal elements $Z, Z' \in \mathfrak{z}$, there exists $Z'' \in \mathfrak{z}$ such that $J_Z J_{Z'} = J_{Z''}$. When $\mathbb{K} = \mathbb{H}$, this last observation follows from the associativity of left multiplication. In the case that $\mathbb{K} = \mathbb{O}$, this follows from the fact that $\dim(\mathfrak{z}) = \dim(\mathfrak{v}) - 1$. Thus we verify that generalized Heisenberg groups are of Heisenberg type and satisfy the J^2 -condition.

For any Lie algebra \mathfrak{n} and corresponding simply connected Lie group N of Heisenberg type, one may construct the Lie algebra $\mathfrak{s} := \mathfrak{n} \oplus \mathfrak{a}$, where \mathfrak{a} is a one-dimensional Lie algebra with inner product. Let \mathfrak{a} be spanned by the unit vector H . The Lie bracket on \mathfrak{s} is determined by the requirements that $[H, X] = \frac{1}{2}X$ and $[H, Z] = Z$ for any $X \in \mathfrak{v}$ and any $Z \in \mathfrak{z}$. We extend the inner products on \mathfrak{n} and \mathfrak{a} to \mathfrak{s} by requiring that \mathfrak{n} is orthogonal to \mathfrak{a} . Proceeding as in [CDKR98, Section 3(a)], one obtains the group $S := \exp(\mathfrak{s})$ as a semidirect product NA , where $A := \exp(\mathfrak{a})$. If we parameterize S via $\mathfrak{v} \times \mathfrak{z} \times \mathbb{R}_+$ by identifying (X, Z, t) with $\exp(X + Z) \exp(\log(t)H) \in S$, then an element $a_t = (0, 0, t) \in A \subset S$ acts on $n = (X, Z, 1) \in N \subset S$ by $a_t(n) = (t^{1/2}X, tZ, t)$. By translating the inner product on \mathfrak{s} , we obtain a left invariant distance on S .

One can then proceed to construct the Siegel-type domain

$$D := \left\{ (X, Z, t) : t - \frac{|X|^2}{4} > 0 \right\} \subset \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}.$$

The domain D can be explicitly identified with S . When the left-invariant distance on S is pulled back to D via this identification, S possesses a simply transitive action on D by affine transformations (see [CDKR98, Section 3(b)]). The group N can be identified with the set

$$\partial D = \left\{ \left(X, Z, \frac{|X|^2}{4} \right) \right\} \subset \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}.$$

Given $n = (x, z) = \exp(X + Z) \in N$, write

$$(3.1) \quad \|n\| := \left(\frac{|X|^4}{16} + |Z|^2 \right)^{1/4}$$

The function $d_N(n, n') := \|n'^{-1}n\|$ defines a distance on N that is invariant under left multiplication. Furthermore, the action of A extends to ∂D such that for any $n, n' \in N$ and $a_t \in A$, we have $d_N(a_t(n), a_t(n')) = t^{1/2}d_N(n, n')$.

When $N = H_{\mathbb{K}}$, the geometry of D is identified by the following result ([CDKR91], [CDKR98], [BTV95, Theorem 4.1.9.A]).

Fact 3.1. *Suppose that D is the Siegel-type domain associated to a Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ of Heisenberg type. Let $m := \dim(\mathfrak{z})$. The algebra \mathfrak{n} satisfies the J^2 -condition if and only if*

- ($m = 0$) the space D is isometric to real hyperbolic space.*
- ($m = 1$) the space D is isometric to complex hyperbolic space.*
- ($m = 3$) the space D is isometric to quaternionic hyperbolic space.*
- ($m = 7$) the space D is isometric to octonionic hyperbolic space.*

In particular, \mathfrak{n} satisfies the J^2 -condition if and only if $\exp(\mathfrak{n})$ is isomorphic to $H_{\mathbb{K}}$.

Assume $N = H_{\mathbb{K}}$, and let $G_{\mathbb{K}}$ denote the isometry group of D . Then $H_{\mathbb{K}}AK$ is an Iwasawa decomposition of $G_{\mathbb{K}}$, where K is the stabilizer of $(0, 0, 1) \in D$ and A is as above. Writing M to denote the centralizer of A in K , [CDKR98, Theorem 7.4] provides the Bruhat decomposition $G_{\mathbb{K}} = (H_{\mathbb{K}}AM) \cup (H_{\mathbb{K}}\sigma H_{\mathbb{K}}AM)$. Here σ is the geodesic inversion of D described in [CDKR98, Section 3(c)].

Via an appropriate Cayley transformation C , the domain D can be identified with the unit ball $B \subset \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$. This Cayley transformation $C : D \rightarrow B$ can be continuously extended to a homeomorphism between the one point compactification of $D \cup \partial D$ and the closed unit ball \overline{B} , where $C(0, 0, 0) = (0, 0, 1)$ and $C(\infty) = (0, 0, -1)$. Via this identification, the action of $G_{\mathbb{K}}$ can be continuously extended to \overline{B} . The stabilizer of $(0, 0, -1) \in \partial B$ is $H_{\mathbb{K}}AM$ (see the proof of [CDKR98, Theorem 7.4]). The stabilizer of both $(0, 0, 1)$ and $(0, 0, -1)$ is AM . In this way we can view $G_{\mathbb{K}}$ as a group acting on $\hat{H}_{\mathbb{K}} = \partial B$. The subgroup $H_{\mathbb{K}}AM$ fixes the point at infinity $\infty \in \hat{H}_{\mathbb{K}}$, and the subgroup AM fixes both the identity element $e \in H_{\mathbb{K}}$ and $\infty \in \hat{H}_{\mathbb{K}}$.

4. PRELIMINARY FACTS AND LEMMAS

The following fact is a special case of [Kra03, Theorem 3.3].

Fact 4.1. *Let G denote a group acting effectively and 2-transitively on a topological sphere. If G is locally compact and σ -compact, then the identity component of G is a simple Lie group isomorphic to either $G_{\mathbb{K}}$ (when $\mathbb{K} = \mathbb{C}, \mathbb{H}$, or \mathbb{O}) or to an index two subgroup of $G_{\mathbb{K}}$ (when $\mathbb{K} = \mathbb{R}$).*

The following technical lemmas will allow us to apply Fact 4.1 to prove Theorem 1.1. We say that a result is *quantitative* provided that the parameters pertaining to the conclusion of a statement quantitatively depend on the parameters pertaining to the assumptions.

Lemma 4.2. *Suppose (X, d) is a locally compact, Ahlfors Q -regular, and Loewner metric space. For any $p \in X$, the spaces (\hat{X}_p, d_p) and (\hat{X}, \hat{d}_p) are quantitatively Ahlfors Q -regular and Loewner, with parameters independent of p .*

Proof. Since metric sphericalization is quasimöbius with parameters independent of p (see [BHX08, Section 3]), the conclusion regarding the Loewner condition follows from [BK05, Theorem 2.7] and [BK02a, Lemma 3.2]. The conclusion regarding Ahlfors regularity is noted in [Fre12, Fact 4.1]. \square

Lemma 4.3. *Suppose (X, d) is an unbounded, proper, Loewner, and inversion invariant bi-Lipschitz homogeneous metric space that is homeomorphic to \mathbb{R}^n ($n \geq 2$). Fix a point $p \in X$. If \mathcal{F} is a family of self-homeomorphisms of (\hat{X}, \hat{d}_p) such that every $f \in \mathcal{F}$ is a homeomorphism from $(X \setminus f^{-1}(\infty), d)$ to $(X \setminus f(\infty), d)$ that is 1-quasiconformal except on a finite set, then the group G generated by \mathcal{F} consists of uniformly θ -quasimöbius self-homeomorphisms of (\hat{X}, \hat{d}_p) , where θ is determined only by the constants associated with (X, d) .*

Proof. Since (X, d) is assumed to be proper, connected, and inversion invariant bi-Lipschitz homogeneous, it follows from [Fre13, Theorem 2.7] that (X, d) is doubling. Therefore, due to [Fre12, Theorem 1.1] we know that (X, d) is Ahlfors Q -regular for $Q \geq n$. Since (X, d) is assumed to be

Loewner, we can apply [HK98, Theorem 3.13] to conclude that (X, d) is linearly locally connected (see [HK98, Section 3.12]). Since X is homeomorphic to \mathbb{R}^n for $n \geq 2$, the fact that (X, d) is Ahlfors Q -regular, Loewner, and linearly locally connected remains true if we remove a finite set from X .

Let g denote an element of $G = \langle \mathcal{F} \rangle$. Then $g : (X \setminus g^{-1}(\infty), d) \rightarrow (X \setminus g(\infty), d)$ is a homeomorphism that is 1-quasiconformal except on a finite set $Z_g \subset X$. Let $x := g^{-1}(\infty)$ and $y := g(\infty)$. Via [BHX08, Proposition 4.1] we conclude that $g : (X_x \setminus Z_g, d_x) \rightarrow (X_y \setminus g(Z_g), d)$ is 1-quasiconformal. Furthermore, this map sends bounded sets to bounded sets, so Lemma 4.2 allows us to apply [HK98, Corollary 4.8] and conclude that $g : (X_x \setminus Z_g, d_x) \rightarrow (X_y \setminus g(Z_g), d)$ is η -quasisymmetric, where η is determined only by the constants associated with (X, d) . Since quasisymmetric maps are quantitatively quasimöbius ([Väi85, Theorem 3.2]), and $\text{id} : (X_x, d) \rightarrow (X_x, d_x)$ is $16t$ -quasimöbius ([BHX08, Lemma 3.1]), we conclude that $g : (X_x \setminus Z_g, d) \rightarrow (X_y \setminus g(Z_g), d)$ is θ' -quasimöbius, where θ' is determined only by the constants associated with (X, d) . By continuity, $g : (X_x, d) \rightarrow (X_y, d)$ is θ' -quasimöbius. Finally, since the identity map $(X, d) \rightarrow (X, \hat{d}_p)$ is $16t$ -quasimöbius, continuity again allows us to conclude that g is a θ -quasimöbius self-homeomorphism of (\hat{X}, \hat{d}_p) , where θ is determined only by the constants associated with (X, d) . \square

Lemma 4.4. *Suppose X is a compact metric space and $G \subset \mathcal{C}(X, X)$ is a group of uniformly quasimöbius self-homeomorphisms. The closure of G is locally compact and σ -compact in the topology of uniform convergence.*

Proof. Let X^3 denote the space of ordered triples from X endowed with the product distance. Since X is compact, X^3 is separable. Let $\{T_i\}$ denote a countable dense subset of X^3 , where $T_i = (x_i, y_i, z_i)$. Given T_i, T_j , and $\varepsilon > 0$, define

$$B_{i,j}(\varepsilon) := \{g \in G : g(T_i) \subset N(T_j; \varepsilon)\},$$

where, for $E \subset X$ and $r > 0$, $N(E; r) := \{x \in X : \text{dist}(x, E) < r\}$. The sets $B_{i,j}(\varepsilon)$ are open in the compact-open topology (and hence in the uniform convergence topology). Furthermore, for fixed i, j there exists $\varepsilon_{i,j} > 0$ such that the set $B_{i,j}(\varepsilon_{i,j})$ is equicontinuous (see [Väi85, Theorem 2.1]). In fact, one can take $\varepsilon_{i,j} := \text{sep}(T_j)/4$, where $\text{sep}(T_j)$ denotes the minimal distance between distinct pairs in T_j . Therefore, by Arzela-Ascoli, $B_{i,j}(\varepsilon_{i,j})$ has compact closure. One can check that $G \subset \cup_{i,j} B_{i,j}(\varepsilon_{i,j})$, and so we conclude that the closure of G is locally compact. Since the collection $\{B_{i,j}(\varepsilon_{i,j})\}$ is countable, the closure of G is σ -compact. \square

5. PROOFS OF THEOREM 1.1 AND THEOREM 1.3

Proof of Theorem 1.1. By assumption, for some $p \in \mathbb{G}$, there exists a 1-quasiconformal metric inversion $\varphi : (\mathbb{G}_p, d_{SR}) \rightarrow (\mathbb{G}_p, d_{SR})$. Up to a conjugation by left-translation, we may assume that $\varphi(e) = \infty$ and $\varphi(\infty) = e$. For any $x \in \mathbb{G}$, write $\varphi_x := \ell_x \circ \varphi \circ \ell_x^{-1}$. Thus φ_x is a metric inversion such that $\varphi_x(x) = \infty$ and $\varphi_x(\infty) = x$.

Let $\{x, x', y, y'\}$ denote a quadruple of points in $\hat{\mathbb{G}}$ such that $x' = y'$ if and only if $x = y$. We consider the two possible cases for such a quadruple:

- Assume that $x \neq y$. If all points are finite, define the map $g := \varphi_{x'} \circ \ell_z \circ \varphi_x : \hat{\mathbb{G}} \rightarrow \hat{\mathbb{G}}$, where $z := \varphi_{x'}^{-1}(y')\varphi_x(y)^{-1} \in \mathbb{G}$. If $x = \infty$ and/or $x' = \infty$, then replace φ_x and/or $\varphi_{x'}$, respectively, with the identity map $\text{id} : \hat{\mathbb{G}} \rightarrow \hat{\mathbb{G}}$.
- Assume that $x = y$ (and so $x' = y'$). If both points x and x' are finite, then replace φ_x and $\varphi_{x'}$ with the identity map on \mathbb{G} . If x and/or x' is the point at infinity, then replace φ_x and/or $\varphi_{x'}$, respectively, with the identity map and replace ℓ_z with the identity map.

In either case we have $g(x) = x'$ and $g(y) = y'$. Furthermore, g is a homeomorphism from $(\mathbb{G} \setminus g^{-1}(\infty), d_{SR})$ to $(\mathbb{G} \setminus g(\infty), d_{SR})$ that is 1-quasiconformal except on a finite set. Let \hat{d}_{SR} denote the sphericalized distance $(\hat{d}_{SR})_e$ on $\hat{\mathbb{G}}$, and write G_* to denote the group of self-homeomorphisms of $(\hat{\mathbb{G}}, \hat{d}_{SR})$ generated by the maps g . By Lemma 4.3 we conclude that the group G_* consists of uniformly quasimöbius self-homeomorphisms of $(\hat{\mathbb{G}}, \hat{d}_{SR})$. By Lemma 4.4 we conclude that the closure

of G_* , denoted by G , is locally compact and σ -compact. Since $\hat{\mathbb{G}}$ is compact and G_* consists of uniformly quasimöbius self-homeomorphisms of $\hat{\mathbb{G}}$, the group G also consists of uniformly quasimöbius self-homeomorphisms.

Note that canonical dilations of \mathbb{G} were not included in the generating set for G_* . However, if Γ denotes the group of canonical dilations of \mathbb{G} , then the closure of $\langle G_*, \Gamma \rangle$ is isomorphic to G . In particular, we may assume that $\Gamma \subset G$. This follows from Lemma 4.4 and [Kra03, Theorem 3.3] because both $\langle G_*, \Gamma \rangle$ and G_* act effectively and 2-transitively on $(\hat{\mathbb{G}}, \hat{d}_{SR})$ by uniformly quasimöbius homeomorphisms.

Given a topological group H , let H° denote the identity component. Via Fact 4.1 we conclude that G° is isomorphic to one of $G_{\mathbb{R}}^\circ$, $G_{\mathbb{C}}^\circ$, $G_{\mathbb{H}}^\circ$, or $G_{\mathbb{O}}^\circ$. In short, G° is isomorphic to $G_{\mathbb{K}}^\circ$. More precisely (see [Kra03, Theorem 3.3 and Proposition 7.1]), there exists an isomorphism $\psi : G^\circ \rightarrow G_{\mathbb{K}}^\circ$ and a homeomorphism $F : (\hat{\mathbb{G}}, \hat{d}_{SR}) \rightarrow (\hat{H}_{\mathbb{K}}, \hat{d}_H)$ such that, for any $g \in G^\circ$ and $x \in \hat{\mathbb{G}}$, we have $F(g(x)) = \psi(g)(F(x))$. Here d_H is defined by (3.1), and \hat{d}_H denotes the sphericalized distance $(\widehat{d_H})_e$. Since $G_{\mathbb{K}}^\circ$ acts two-transitively on $(\hat{H}_{\mathbb{K}}, \hat{d}_H)$, we may assume $F(e) = e$ and $F(\infty) = \infty$.

Given $t > 0$, the map $\psi(\delta_t) \in G_{\mathbb{K}}^\circ$ fixes the set $\{e, \infty\}$. Therefore, $\psi(\delta_t) \in AM$. It follows that there exists $a_{s(t)} \in A$ and $m_t \in M$ such that $\psi(\delta_t) = a_{s(t)}m_t$. Here $s : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a function such that, for any $s, r \in \mathbb{R}_+$, we have $s(rt) = s(r)s(t)$. Since, for any $x \in \mathbb{G}$, we have $F(\delta_r(x)) = \psi(\delta_r)(F(x))$, it also follows that $\lim_{t \rightarrow +\infty} s(t) = +\infty$. Since A commutes with M , we note that $\psi(\delta_t^{-1}) = a_{1/s(t)}m_t^{-1}$.

Given $g \in \mathbb{G}$, the map $\psi(\ell_g)$ fixes the point at infinity in $\hat{H}_{\mathbb{K}}$. Therefore, $\psi(\ell_g) \in H_{\mathbb{K}}AM$, and there exist $h \in H_{\mathbb{K}}$, $a_r \in A$, and $m \in M$ such that $\psi(\ell_g) = \ell_h a_r m$. Combining this with the previous paragraph, we have $\psi(\delta_t^{-1} \ell_g \delta_t) = a_{1/s(t)}m_t^{-1} \ell_h a_r m a_{s(t)}m_t$. We then observe that

$$\begin{aligned} a_{1/s(t)}m_t^{-1} \ell_h a_r m a_{s(t)}m_t &= a_{r/s(t)}m_t^{-1} \ell_{[a_{1/r}(h)]} a_{s(t)}m m_t \\ &= a_r m_t^{-1} \ell_{[a_{1/(rs(t))}(h)]} m m_t \end{aligned}$$

Since M is compact and $a_{1/(rs(t))}(h) \rightarrow e \in H_{\mathbb{K}}$ as $t \rightarrow +\infty$, there exists $m' \in M$ such that, up to a subsequence,

$$a_r m_t^{-1} \ell_{[a_{1/(rs(t))}(h)]} m m_t \rightarrow a_r m'^{-1} m m'$$

as $t \rightarrow +\infty$. Here the convergence is uniform on compact subsets of $H_{\mathbb{K}}$. On the other hand, the map $\delta_t^{-1} \ell_g \delta_t = \ell_{[\delta_t^{-1}(g)]}$ is locally uniformly convergent to the identity map of \mathbb{G} . Since $\psi : G^\circ \rightarrow G_{\mathbb{K}}^\circ$ is an isomorphism and both groups act effectively, $a_r m'^{-1} m m' = \text{id}$ as a map of $(H_{\mathbb{K}}, d_H)$. Via the Bruhat decomposition, this implies that $a_r = \text{id}$ and $m'^{-1} m m' = \text{id}$, and so $m = \text{id}$. Therefore, $\psi(\ell_g) = \ell_h$. Since \mathbb{G} acts simply transitively on itself by left translations, we conclude that $\psi(\mathbb{G}) = H_{\mathbb{K}}$.

In order to prove the reverse implication, suppose \mathbb{G} is a generalized Heisenberg group equipped with a sub-Riemannian distance. Due to the comparability of the sub-Riemannian distance with the distance given by (3.1), by [CDKR91, Theorem 4.2], the map σ satisfies the definition of a metric inversion and is quasiconformal on \mathbb{G}_e . By (the proof of) [CDKR91, Theorem 5.1] the Riemannian differential of σ , when restricted to the horizontal distribution, is the composition of a dilation and an isometry at every point in \mathbb{G}_e . It then follows from [CC06, Lemma 3.4 and Corollary 7.2] that the map σ is 1-quasiconformal on \mathbb{G}_e . \square

The above proof can be compared with the methods appearing in [BS13]. In [BS13], the ideal boundaries of rank one symmetric spaces are characterized via the notion of *space inversions*. Such inversions are Möbius involutions that satisfy several additional properties related to the Möbius structure of a metric space. While metric inversions on a bi-Lipschitz homogeneous space can be used to construct analogues to space inversions, such constructions need not possess all the properties required to apply the innovative techniques behind the proof of [BS13, Theorem 1.1].

Proof of Theorem 1.3. This result is obtained by combining results from [BK02b], [Kin13], and [Fre13]. We first consider the case that (X, d) is unbounded. Suppose (X, d) is inversion invariant bi-Lipschitz homogeneous with respect to a collection of uniformly L -bi-Lipschitz self-homeomorphisms \mathcal{F} and an L -metric inversion φ based at some point $p \in X$. It follows from [Fre13, Theorem 2.7] that (X, d) is doubling. Therefore, by [Fre12, Theorem 1.1], (X, d) is Ahlfors n -regular. As noted in [Fre12, Fact 4.1], for any point $q \in X$, the sphericalized space (\hat{X}, \hat{d}_q) is also Ahlfors n -regular, with regularity constant depending only on the doubling and homogeneity constants for (X, d) .

Fix $q \in X$ and set $\delta := (16L^3M(1+M))^{-1}$, where $1 \leq M < +\infty$ is the quasi-self-similarity constant given by [Fre13, Theorem 2.7]. Note that M depends only on L . Let $\{x_1, x_2, x_3\}$ denote a triple of distinct points in \hat{X} . We claim that there exists a strongly quasimöbius homeomorphism $g : (\hat{X}, \hat{d}_q) \rightarrow (\hat{X}, \hat{d}_y)$ such that, for $i \neq j$, we have $\hat{d}_q(g(x_i), g(x_j)) \geq \delta$. When $x_1 \neq \infty$, there exists $f \in \mathcal{F}$ such that $f(x_1) = p$. When $x_1 = \infty$, set $f := \varphi$. In either case, $\varphi \circ f(x_1) = \infty$ and we define $y := \varphi \circ f(x_2)$, $z := \varphi \circ f(x_3)$. Consider the sphericalized space (\hat{X}, \hat{d}_y) . By [Fre13, Theorem 2.7], there exists a map $h : (X, d) \rightarrow (X, d)$ such that $h(y) = y$ and, for all $u, v \in X$, we have $d(h(u), h(v)) \simeq_M C d(u, v)$, where $C := d(y, z)^{-1}$. It follows from (2.1) that

$$\hat{d}_y(h(y), h(z)) \geq \frac{1}{4} \cdot \frac{d(h(y), h(z))}{1 + d(h(y), h(z))} \geq \frac{1}{4M} \cdot \frac{C d(y, z)}{1 + MC d(y, z)} = \frac{1}{4M(1+M)}$$

We also find that

$$\begin{aligned} \hat{d}_y(h(\infty), h(z)) &= \hat{d}_y(\infty, h(z)) \geq \frac{1}{4} \cdot \frac{1}{1 + d(h(y), h(z))} \\ &\geq \frac{1}{4} \cdot \frac{1}{1 + MC d(y, z)} = \frac{1}{4(1+M)}. \end{aligned}$$

Finally, we note that $\hat{d}_y(y, \infty) \geq 1/4$. Therefore, $h \circ \varphi \circ f$ maps $\{x_1, x_2, x_3\}$ to a $(4M(1+M))^{-1}$ separated triple in (\hat{X}, \hat{d}_y) . By [BHX08, Lemma 3.2], there exists a $4L^3$ -bi-Lipschitz homeomorphism $k : (\hat{X}, \hat{d}_y) \rightarrow (\hat{X}, \hat{d}_q)$. Therefore, for any triple of distinct points $\{x_1, x_2, x_3\} \subset (\hat{X}, \hat{d}_q)$, there exists a map of the form $g = k \circ h \circ \varphi \circ f$ such that g is a Kt -quasimöbius self-homeomorphism of (\hat{X}, \hat{d}_q) that maps $\{x_1, x_2, x_3\}$ to a δ -separated triple. Here K depends only on L .

By [Kin13, Theorem 5.1], we conclude that (\hat{X}, \hat{d}_q) is strongly quasimöbius equivalent to \mathbb{S}^n . To justify this application of [Kin13, Theorem 5.1], note that this theorem follows from results in [BK02b, Section 5]. These results are proved under the assumption of a group action by uniformly quasimöbius homeomorphisms. However, as stated at the beginning of [BK02b, Section 5], these results also hold under the weaker assumption that triples can be uniformly separated by uniformly quasimöbius maps.

Since strongly quasimöbius maps between bounded spaces are bi-Lipschitz (see [Kin13, Remark 3.2]), we find that (\hat{X}, \hat{d}_q) is bi-Lipschitz equivalent to \mathbb{S}^n . Finally, by [BHX08, Lemma 3.2 and Proposition 3.4], (X, d) is bi-Lipschitz equivalent to \mathbb{R}^n .

To finish the proof, we consider the case that (X, d) is bounded. Given any point $p \in X$, the space (X_p, d_p) is unbounded and remains proper, connected, and inversion invariant bi-Lipschitz homogeneous. To verify that (X_p, d_p) remains inversion invariant bi-Lipschitz homogeneous, note that the metric inversion of (X_p, d_p) at any point $q \in X_p$ is bi-Lipschitz equivalent to (X_q, d_q) via the identity map. We conclude as above that (X_p, d_p) is bi-Lipschitz equivalent to \mathbb{R}^n . By [BHX08, Lemma 3.2 and Proposition 3.5], (X, d) is bi-Lipschitz equivalent to \mathbb{S}^n . \square

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